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Longshore motion generated on beaches by obliquely incident bores

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Numerical solutions of the two-dimensional shallow-water equations are found for motion on a sloping beach generated by a single bore and by a periodic succession of bores, both incident at small angles. For the latter, periodic longshore motion can always be found if bottom friction (described here by the Chezy law) in included. The timescale of the development of longshore motion is also considered.

1. Introduction

In many circumstances, longshore motion on a beach, shoreward of wave breaking, is generated by obliquely incident waves. The existence of longshore currents has long been known, and their importance, particularly their role in sediment transport, recognized. Komar (1976) discusses various existing predictions and correlations.

The main difficulties in accurate mathematical modelling of longshore currents lie in the use of appropriate models for wave motion in shallow water and for the effect of viscous dissipation due to bottom friction. After waves have broken, there is normally a region, the surf zone, where wave fronts are steep and turbulent but where surface slopes are otherwise gentle. Such wave fronts are well described by the mathematical model of a turbulent bore. Longuet-Higgins (1970*a*, *b*) shows that the dissipation of wave energy within the surf zone provides, in a distinctive way, the driving stress for longshore motion. In this work, such dissipation occurs as turbulent dissipation at bore fronts.

In this paper we shall extend existing models of the propagation of normally incident bores in order to describe longshore motion generated by waves incident at a small angle on a plane beach. Although a bore is commonly the result of a wave having broken, we make no attempt to describe the breaking process or to connect surf-zone motion with wave motion before breaking. The arrival of a bore in the area of beach in which solutions are found is represented only by a sudden change in flow properties at the edge of that area.

In § 2 we develop the equations used to describe the flow. We use the two-dimensional form of the finite-amplitude shallow-water equations, with the inclusion of a term representing bottom friction. We show that, if the angle of incidence at breaking is small, as offshore depth refraction often ensures is the case, equations for water depth and onshore velocity may be solved independently of longshore velocity. This latter may then be found by separately solving the longshore momentum equation.

In § 3 we consider the longshore motion generated by a single bore incident at a small angle to a plane beach. Though this problem is of less practical significance than the periodic motion studied later, it serves as a simple illustration of a solution. We also, in this problem, omit any representation of bottom friction, in order to allow

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comparisons and checks with exact inviscid results. The motion generated by a normally incident bore travelling up a sloping beach into water at rest is considered by Hibberd & Peregrine (1979), who solve the shallow-water equations using a Lax–Wendroff finite-difference scheme, which automatically allows for the existence and propagation of bores; and by Keller, Levine & Whitham (1960), who use a simple but accurate approximate rule, suggested by Whitham (1958). Further analytical work on the motion of bores is summarized by Meyer & Taylor (1972). In the present work we use the solutions of Hibberd & Peregrine for normally incident bores to describe the onshore motion, and we find the corresponding weak longshore motion.

The main features of the onshore motion are described by Hibberd and Peregrine. We shall omit, as do Hibberd & Peregrine, the influence of the reflected bore, formed, in mathematical terms, by the convergence of receding characteristics.

In considering motion on a real beach, a more realistic problem is that of finding the motion generated by a succession of incident bores, each travelling up the beach into the backwash of the preceding one. Since the inclusion of bottom friction is then necessary, we adopt the Chezy law to model, albeit crudely, the bottom shear stress. The onshore motion generated by a succession of normally incident bores is considered by Packwood (1980), who finds periodic solutions when the incident waves are periodic. In § 4, we consider the longshore motion generated by periodic bores incident at a small angle, with the solutions of Packwood serving as a basic onshore flow. By explicitly seeking a periodic solution, and owing to the decoupling of the longshore and the onshore problems, we find a simple, characteristic-tracing solution method. We calculate an example of such a solution, and compare results with those of Longuet-Higgins (1970*a*,*b*), who uses linear long-wave theory to find a simple expression for the longshore current in terms of easily measured parameters of the incident waves.

The formulation of the shallow-water equations used here is appropriate only for problems in which the motion is completely regular, as discussed in §2. In practice such conditions often do not occur (see e.g. Bowen & Guza 1978). Interaction with edge waves and wave groups formed by interference between two or more trains of incident waves are possible mechanisms by which the heights of breaking waves may vary. Though the solutions found here are valid only for strictly periodic incident waves, we may suppose that, if modulations in incident waves are sufficiently slow, such solutions will remain valid. We shall estimate, in §4, how fast we expect such a solution to react to changes in the incident waves. In a further paper (Ryrie 1982) we consider the more general problem of how variations in incident waves may affect longshore motion.

2. Mathematical model

2.1. *Shallow-water equations*

The assumption that water motion within the surf zone is such that, except near a bore, the lengthscale of variations in flow properties is much greater than the depth leads to the use of the finite-amplitude shallow-water equations (see e.g. Peregrine 1972). The equations for conservation of mass and momentum may be written, using asterisks to denote dimensional quantities, as

$$\frac{\partial h^*}{\partial t^*}+\frac{\partial}{\partial x_i^*}(h^*u_i^*) = 0, \tag{1}$$

$$\frac{\partial}{\partial t^*}(\rho^*h^*u_i^*)+\frac{\partial}{\partial x_j^*}(\rho^*h^*u_i^*u_j^*)+\rho^*g^*h^*\frac{\partial\eta^*}{\partial x_i^*}+B_i^* = 0, \tag{2}$$

where $h^*(x_i^*, t^*)$ denotes water depth, $u_i^*(x_i^*, t^*)$ denotes depth-averaged water velocity, B_i^* denotes stress due to bottom friction, and $\eta^*(x_i^*, t^*)$ denotes the surface elevation above some reference level.

We consider a regular plane beach whose contours are straight and parallel to the x_2 direction: x_1^* and x_2^* are therefore coordinates respectively towards and parallel to the shoreline. We write the water depth as

$$h^*(x_i^*, t^*) = -sx_1^* + \eta^*(x_i^*, t^*), \tag{3}$$

where s denotes the beach slope, assumed to be constant.

In order to make quantities in the above expressions dimensionless we use the scheme of Hibberd & Peregrine (1979). This scheme has the advantage that any explicit dependence on s, except in the friction term, is removed, if s is small: we assume that s is sufficiently small that we need not distinguish between distances measured along the beach and along the horizontal. The dimensionless variables used are

$$\left.\begin{aligned} x_i = \frac{sx_i^*}{H^*}, \quad (h, \eta) = \frac{(h^*, \eta^*)}{H^*}, \quad u_i = \frac{u_i^*}{(g^*H^*)^{\frac{1}{2}}}, \\ t = t^*s\left(\frac{g^*}{H^*}\right)^{\frac{1}{2}}, \quad B_i = \frac{B_i^*}{\rho^*g^*H^*s}, \end{aligned}\right\} \tag{4}$$

where H^* is a reference depth, such as the depth of water at the seaward boundary of the area of beach being considered.

In terms of these variables (1) and (2) become

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i}(hu_i) = 0, \tag{5}$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}(h + x_1) + \frac{B_i}{h} = 0. \tag{6}$$

We shall assume that the solutions to be found of these equations travel along the beach at a speed $c^*/\sin\theta$, where c^* and θ are respectively the phase speed and angle of incidence of the incident waves: the quantity $c^*/\sin\theta$ remains constant through the surf zone, by Snell's law. For this assumption to be valid, wave properties seaward of the area of beach being considered must be periodic both in space and in time. We need therefore to assume that all waves originate offshore with the same initial conditions, so that the wave train sufficiently far offshore has straight parallel crests. We need also to assume that the offshore topography does not vary in the x_2 direction, so that all points along a wave crest traverse the same topography up to breaking. Waves generated by, for instance, a point source at a finite depth do not satisfy these assumptions.

Following these assumptions, we define a new independent variable t', referred to as pseudotime, by

$$t'^* = t^* - \frac{\sin\theta_B}{c_B^*}x_2^* \tag{7}$$

where subscript B denotes conditions at the breaking point: the origin of the coordinate t' moves along the beach with the solution.

We also restrict our attention to waves incident at a small angle θ_B to the beach. The lengthscale of variations in the x_2 direction is therefore much greater than that in the x_1 direction. Since waves approaching a beach from deep water are, in most circumstances, refracted towards the beach so that θ_B is small, this restriction is not severe.

Accordingly, we scale x_2 by defining

$$x_2' = \frac{x_2}{\epsilon},$$

where

$$\epsilon = \frac{\sin\theta_B}{c_B^*}(g^*H^*)^{\frac{1}{2}}$$

is a small parameter, chosen to make variations in x_2' of order unity. We also scale u_2 by defining

$$u_2' = \frac{u_2}{\epsilon},$$

choosing the scaling factor in order to ensure that the resulting equations of motion remain as general as possible.

Equations (5) and (6) become, after replacing $\partial/\partial t$ and $\partial/\partial x_2$ by $\partial/\partial t'$ and $-\partial/\partial t'$ respectively,

$$h_{t'} + (hu_1)_{x_1} = \epsilon^2 (hu_2')_{t'}, \tag{8}$$

$$u_{1t'} + u_1 u_{1x_1} + h_{x_1} + 1 + \frac{B_1}{h} = \epsilon^2 u_2' u_{1t'}, \tag{9}$$

$$\epsilon u_{2t'} + \epsilon u_1 u_{2x_1}' - \epsilon h_{t'} + \epsilon \frac{B_2}{h} = \epsilon^3 u_2' u_{2t'}'. \tag{10}$$

Equations (8) and (9), with ϵ and B_1 set to zero, are the inviscid finite-amplitude shallow-water equations, for onshore motion only, as used by Hibberd & Peregrine (1979). The use of pseudotime and the assumption of longshore regularity means that (8)–(10) describe the motion at a fixed point along the beach for varying time, as solutions move bodily along the beach.

We shall find, in considering periodic bores, that it is necessary to include some representation of the effect of bottom friction in order to find physically realistic solutions. We adopt, for this purpose, the commonly used Chézy friction law, according to which the bottom-friction stress is

$$B_i^* = C\rho^*(u_j^* u_j^*)^{\frac{1}{2}} u_i^*,$$

where C is a dimensionless constant. There are various estimates of the value of the friction coefficient C likely to apply to unsteady motion over a sandy bottom. Longuet-Higgins (1970*a*, *b*) takes an estimate, based on Nikuradse's experiments on sand-roughened pipes, of about 0·007 for sand grains with a diameter of 1 mm. Meyer (1969) suggests an order of magnitude of about 0·01 for C. Grant & Madsen (1979) estimate bottom-friction stress for oscillating motion over a rough bottom by taking a turbulent boundary-layer model for flow near the bottom: for a typical beach they find C also of order 0·01.

In terms of the dimensionless variables used here,

$$B_i = f(u_j u_j)^{\frac{1}{2}} u_i,$$

where $f = C/s$.

The solution of (8)–(10) for an arbitrary value of ϵ presents a considerable problem. We shall considerably simplify it, however, by using only the terms in each equation of leading order in ϵ, thereby finding solutions to zero order in ϵ for the zero-order quantities h and u_1 and to first order for the first-order quantity u_2. We therefore use the equations

$$h_t + (hu)_x = 0, \tag{11a}$$

$$u_t + uu_x + h_x + 1 + \frac{f|u|u}{h} = 0, \tag{11b}$$

$$v_t + uv_x - h_t + \frac{f|u|v}{h} = 0, \tag{12}$$

where we have, for convenience, replaced x_1 and x_2' by x and y; u_1 and u_2' by u and v, and t' by t, referring to the latter as the time variable. We see that (12) is decoupled from (11a, b), and is a perturbation equation for $v(x, t)$, once $h(x, t)$ and $u(x, t)$ are known, as solutions of (11), or indeed by any other means. We shall refer to the separate problems of solving (11) and (12) as respectively the onshore problem and the longshore problem. In physical terms, the decoupling of the longshore momentum equation means that we neglect any effect on the onshore motion of interaction between onshore and longshore motion. This neglect is justifiable when the omitted terms of (8)–(10) are indeed small compared with the terms retained: we shall, in §4, estimate the likely range of ϵ and hence of θ_B, for which their omission is appropriate.

The description that we shall use of wave fronts, as turbulent bores, means that the shallow-water equations must be supplemented by bore relations, regarded as internal boundary conditions, between the physical quantities h, u and v on either side of a bore. These relations, which ensure continuity across a bore of mass flux, momentum flux normal to the bore and of the velocity component parallel to the bore, are derived by, for instance, Lamb (1932).

In our notation, they may be written

$$-U[h] + [hu] = 0, \tag{13a}$$

$$-U[hu] + [hu^2 + \tfrac{1}{2}h^2] = 0, \tag{13b}$$

$$-U[hv - \tfrac{1}{2}h^2] + [huv] = 0, \tag{13c}$$

where U is the speed of the bore, and [ ] denotes the jump in a quantity across a bore.

These relations are those which apply to uniform flow on either side of a hypothetical point discontinuity: any effects due to the structure of the bore or to changing depth near the bore are not included.

2.2. *Characteristic form*

The shallow-water equations are hyperbolic, and their equivalent characteristic form can, in some circumstances, be useful, particularly if the problem being considered is such that Riemann invariants can be found. Analysis of (11), (12) by standard methods (e.g. Courant & Hilbert 1962) yields an equivalent form defining three families of characteristics:

$$(u \pm 2c + t)_\sigma + f\frac{|u|\,u}{h} = 0 \quad \text{along} \quad \frac{dx}{dt} = u \pm c, \tag{14}$$

$$(v - h - x - \tfrac{1}{2}h^2)_\sigma - f\frac{|u|\,u^2}{h} + f\frac{|u|\,v}{h} = 0 \quad \text{along} \quad \frac{dx}{dt} = u, \tag{15}$$

where $c = h^{\frac{1}{2}}$; in each case a subscript σ denotes differentiation along the corresponding characteristic whose local slope dx/dt has been found; i.e.

$$\frac{\partial}{\partial \sigma} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x}.$$

If bottom friction is neglected, all three families of characteristics have Riemann invariants. Equation (14) then defines the well-known characteristics for one-dimensional inviscid motion: we refer to these as C^+ and C^- characteristics, depending upon the sign taken.

Equation (15) defines a new family of characteristics, referred to as C^0 characteristics, which carry information about the longshore motion. The paths of these characteristics are fluid-particle paths: the quantity $v-h-x-\frac{1}{2}u^2$ is constant along these paths, and is thus carried along with a fluid particle, if there is no friction. We introduce notation for the characteristic variables as

$$\begin{aligned} \alpha &= u+2c+t, \\ -\beta &= u-2c+t, \\ \gamma &= v-h-x-\tfrac{1}{2}u^2, \end{aligned}$$

so that α, β and γ are Riemann invariants if there is no dissipation.

When bottom friction is included, however, no set of characteristics possesses a Riemann invariant: the effect of frictional dissipation is to change continuously, along a characteristic path, the value of the characteristic variable, which would otherwise remain constant.

Although we have carried out the above analysis by assuming the Chézy friction law, the paths of all three characteristic families, and the corresponding characteristic variables, are independent of the expression used for the bottom-friction stress, so long as it does not involve derivatives of u and h. The major qualitative features of the motion we are considering are then independent of the form of friction law used.

The influence of longshore motion on the C^+ and C^- characteristics may be found by carrying out the derivation of the characteristic form ((14), (15)) with the inclusion of higher-order terms in ϵ, from (8)–(10). After doing so, we find that the $O(\epsilon^2)$ terms that appear in (14) are such as to make this equation non-integrable, even if friction is excluded: the C^+ and C^- characteristics do not therefore have Riemann invariants in this case. This does not apply to (15): the C^0 characteristics do have Riemann invariants when $O(\epsilon^2)$ terms are included, if friction is neglected.

3. A single uniform bore

In this section, we apply the shallow-water equations (11), (12) to find the longshore motion generated by a single bore, incident at a small angle, during both the travel of the bore up the beach and the subsequent run-up and backwash. For the reasons described in §1, we neglect bottom friction. We take the undisturbed shoreline and the seaward boundary, at which the undisturbed dimensionless depth is 1, to be at $x = 0$ and $x = -1$ respectively: the bed has a flat bottom in $x < -1$. The incident bore is uniform in the sense that, as it approaches from $x < -1$, flow properties behind it are constant. It therefore represents the simplest physically realizable bore, a boundary between two regions of constant state. The bore reaches $x = -1$ at the moment $t = 0$.

Assuming that the incident bore is subcritical, as bores on a beach usually are, the characteristics of the motion are as sketched in figure 1. In $t < 0$, advancing C^+ characteristics arrive at the bore, and receding C^- and advancing C^0 characteristics leave it. Region I, in which the sloping beach has no influence, is of constant state, with straight characteristics. Since there is no bottom slope here, the forms of the Riemann invariants are different from those found in §2: their constant values are

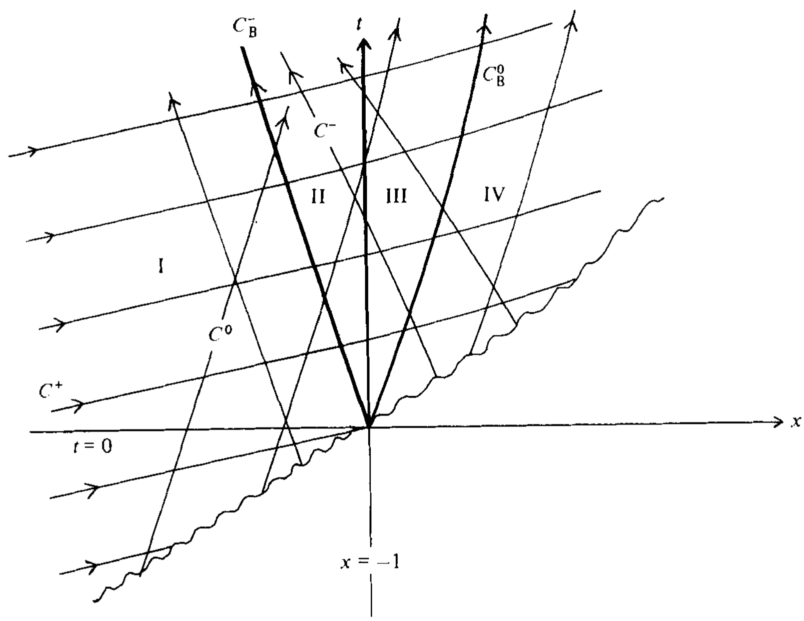

FIGURE 1. Paths of C^+, C^- and C^0 characteristics, for a single bore, in (x, t)-space. The path of the bore is shown by the wavy line; heavy lines separate regions I–IV of the solution, as described in the text.

now

$$\alpha_1 = u + 2c, \tag{16a}$$

$$-\beta_1 = u - 2c, \tag{16b}$$

$$\gamma_1 = v - h - \tfrac{1}{2}u^2 \tag{16c}$$

on C^+, C^- and C^0 characteristics respectively.

The receding C^- characteristic, denoted by C_B^-, which leaves $x = -1$ at $t = 0$, bounds region II, within which the influence of the beach is first felt. Region II is also bounded by $x = -1$, so that the form (16) of the Riemann invariants still holds within it; the advancing C^+ characteristics carry the value α_1 up to $x = -1$. This is the boundary condition used on $x = -1$ by Hibberd & Peregrine.

The C^0 characteristics leaving the bore in region I carry the value γ_1 to the boundary $x = -1$: we use this as the boundary condition for the longshore problem, since it is that necessary for consistency with uniform motion in region I.

The C^0 characteristic, denoted by C_B^0, which leaves the bore at $x = -1$, bounds region III, within which the sloping beach has no influence on the longshore motion, and where $\gamma = v - h - \frac{1}{2}u^2 - x$ is constant and equal to $\gamma_1 + 1$. Shoreward of C_B^0, in region IV, the influence of the beach slope is felt, and values of γ are those immediately behind the bore, as determined by the bore relations. The specified distribution of v on $x = -1$ therefore affects the solution only within region III.

We therefore seek solutions of (11), (12), with boundary conditions

$$h(x, 0) = -x, \tag{17a}$$

$$u(x, 0) = v(x, 0) = 0, \tag{17b}$$

$$u + 2c = \text{constant} \quad \text{on} \quad x = -1, \tag{17c}$$

$$v - h - \tfrac{1}{2}u^2 = \text{constant} \quad \text{on} \quad x = -1. \tag{17d}$$

We find solutions, for $h(x,t)$ and $u(x,t)$, of (11) by the numerical scheme of Hibberd & Peregrine (1979), and we develop a similar scheme to solve (12) for $v(x,t)$.

For hyperbolic equations in conservation form, and when shocks, or bores, may be present, a Lax–Wendroff finite-difference scheme, which ensures that the bore relations are satisfied whenever a bore appears, is particularly suitable: such schemes are described by Richtmyer & Morton (1967). Equation (12), without the friction term, may be written in conservation form as

$$R_t + S_x = 0,$$

where

$$R(x,t) = hv - \tfrac{1}{2}h^2,$$
$$S(x,t) = huv.$$

We find solutions on a space–time grid, with step length Δx and time-step Δt: $\Delta x = 0{\cdot}01$ and $\Delta t = 0{\cdot}004$ were used. The finite-difference scheme used was

$$R_j^{n+1} = R_j^n - \tfrac{1}{2}\lambda(S_{j+1}^n - S_{j-1}^n) + \tfrac{1}{2}\lambda^2(g_j^n - g_{j-1}^n) + \tfrac{1}{2}\lambda(d_j^n - d_{j-1}^n), \tag{18}$$

where

$$g_j^n = \tfrac{1}{2}(u_{j+1}^n + u_j^n)\,(S_{j+1}^n - S_j^n),$$
$$d_j^n = |u_{j+1}^n - u_j^n|\,(R_{j+1}^n - R_j^n),$$
$$\lambda = \Delta t/\Delta x.$$

The last term of (18) is the so-called artificial-viscosity term, which is commonly included in Lax–Wendroff methods when applied to nonlinear problems. Its effect is to attenuate a spurious numerical effect, the appearance of high-frequency oscillations which may develop near bores: elsewhere, this term is only of the same order as the truncation error and has very little effect. A similar term was included in the scheme of Hibberd & Peregrine.

The linear stability criterion for the scheme of (18) without the artificial-viscosity term is the Courant condition

$$\frac{\Delta t}{\Delta x} \leqslant \frac{1}{|u_{\mathrm{m}}|},$$

where u_{m} is the maximum velocity found.

The scheme of (18) allows the calculations of v at all grid points, given values of h and u at all points, except for grid points at the shoreline. For such points, a simpler first-order one-sided finite-difference approximation to the characteristic equation was used, and produced satisfactory results.

3.1. *Analytic solution*

We may use an existing model of onshore motion during run-up to gain some expectation of the longshore motion during this phase of the solution. Keller *et al.* (1960) show that, while bore height tends to zero as the bore approaches the shoreline, the bore speed U and the shoreward particle velocity u immediately behind the bore tend to the same limit, u_{e}, say. The value of u_{e} determines the motion at and close to the shoreline during run-up and backwash, though the value predicted by the point discontinuity model of Keller *et al.* may not in practice be attained, since the effects of bore structure are likely to be significant when the distance of the bore from the shoreline is comparable to bore width. The dependence of shoreline motion only on

u_e allows Shen & Meyer (1963) to find solutions of the shallow-water equations valid near the shoreline during run-up. In our notation, these are:

$$h(\xi,\tau) = \frac{\xi^2}{9\tau^2}, \tag{19a}$$

$$u(\xi,\tau) = u_e - \tau - \frac{2\xi}{3\tau}, \tag{19b}$$

where

$$\xi = x - x_s.$$

τ denotes time since start of run-up, and x_s denotes the position of the shoreline, given by

$$x_s = u_e\tau - \tfrac{1}{2}\tau^2. \tag{20}$$

We may extend this to find the longshore velocity $v(\xi,\tau)$ near the run-up tip by solving the longshore momentum (12), with $f = 0$, and h and u given by (19). We find the solution to be

$$v(\xi,\tau) = v_s + \frac{1}{3}\left(\frac{\xi^2}{\tau^2} - \xi - 2u_e\frac{\xi}{\tau}\right) \tag{21}$$

for some constant v_s. This indicates that the value of v at the shoreline $\xi = 0$ is constant with time, and is equal to v_s. This is consistent with the observation of Shen & Meyer that the motion of the shoreline is that of a particle moving freely under gravity. We may find v_s by noting that the shoreline, defined by $h_s = 0$, $u_s = dx_s/dt$, where a subscript s denotes evaluation at the shoreline, is a C^0 characteristic, along which

$$\gamma_s = v_s - \tfrac{1}{2}u_s^2 - x_s = \text{constant}.$$

After using (20) and the bore relations (13), we find

$$v_s = u_e^2. \tag{22}$$

Equations (19) and (21) now express the motion near the shoreline, in terms only of the final bore velocity u_e. After the formation of the backwash bore, however, the shoreline is no longer a coincidence of C^+ and C^- characteristics and no longer moves only under gravity, so that (22) no longer holds.

The derivation of (19) and (21) neglects the influence of motion behind the bore before it collapses. One result of this motion is the creation of vertical vorticity: in physical (x, y)-space the bore crest is curved, and its varying angle of incidence causes the flow behind it to be rotational. The velocity field during run-up is therefore also rotational. The velocity field of (19) and (21) is, however, irrotational, and we do not expect it to provide an accurate prediction of longshore motion except near the shoreline itself.

3.2. *Description of results*

We show results for a bore of initial height 0·5. The calculation proceeds until a new constant water level, with negligible onshore motion, is reached. Solutions to the onshore problem are shown by Hibberd & Peregrine, for an initial bore of height 0·6: results for a bore of height 0·5 are similar.

Figure 2 shows instantaneous longshore velocity profiles at various times. The shoreline velocity reaches a maximum at about $t = 2{\cdot}0$, and then remains nearly constant until the backwash bore forms at about $t = 4{\cdot}5$. The computed values of v

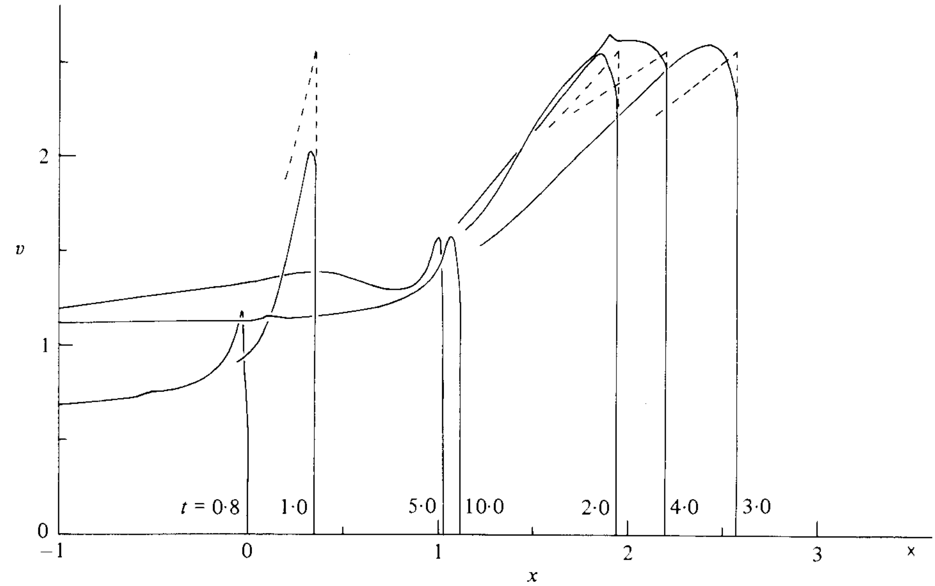

FIGURE 2. Instantaneous profiles of longshore velocity v versus x at times during the computation, as shown, for a single bore, of initial height 0·5. ---, local shoreline solution, from (21) with $u_e = 1{\cdot}6$.

near the shoreline do not increase monotonically, but are rounded off over the last few grid steps. This is an effect of finite resolution on the modelling of the bore collapse: the effect is carried along the C^0 characteristics and remains close to the shoreline.

For comparison, the local shoreline solution of (21) is also shown, using the value $u_e = 1{\cdot}6$ found from the solution to the onshore problem. Reasonable agreement is found, once the initial shoreline acceleration has taken place.

The backwash bore substantially decreases the velocity near the shoreline, which is rapidly brought to rest. The motion then settles to a new steady state, with a constant value v_1 of v except shoreward of the characteristic C^0_B, where the vorticity caused by the bore's oblique travel up the beach remains. The boundary condition at $x = -1$, that $v - h - \frac{1}{2}u^2$ remain constant there, may be used to find the expected value of v_1, once the new constant water level is known: this level may itself be deduced from the boundary condition to the onshore problem. We find $v_1 = 1{\cdot}14$ to be the expected value; this agrees closely with the results of the computations. If the effect of the reflected bore were included, the eventual steady value would be $v_1 = 1{\cdot}09$, only slightly less. Peregrine (private communication) shows that the reflected bore does indeed form, offshore of $x = -1$, at a time within our computations.

A further check on accuracy was made by checking that the total momentum within the computation area was always equal to the time-integrated flux of momentum through the seaward boundary.

In figure 3 we show contours of longshore velocity and fluid particle paths (which are C^0 characteristics) for an initial bore of height 0·5. The characteristic C^0_B, which leaves $x = -1$ at $t = 0$, lies always within the computation area and takes an eventual position close to the initial shoreline. This justifies the use of the boundary condition $\gamma =$ constant on $x = -1$, and shows that longshore motion during a large part of the run-up and backwash is uninfluenced by longshore motion at the seaward boundary.

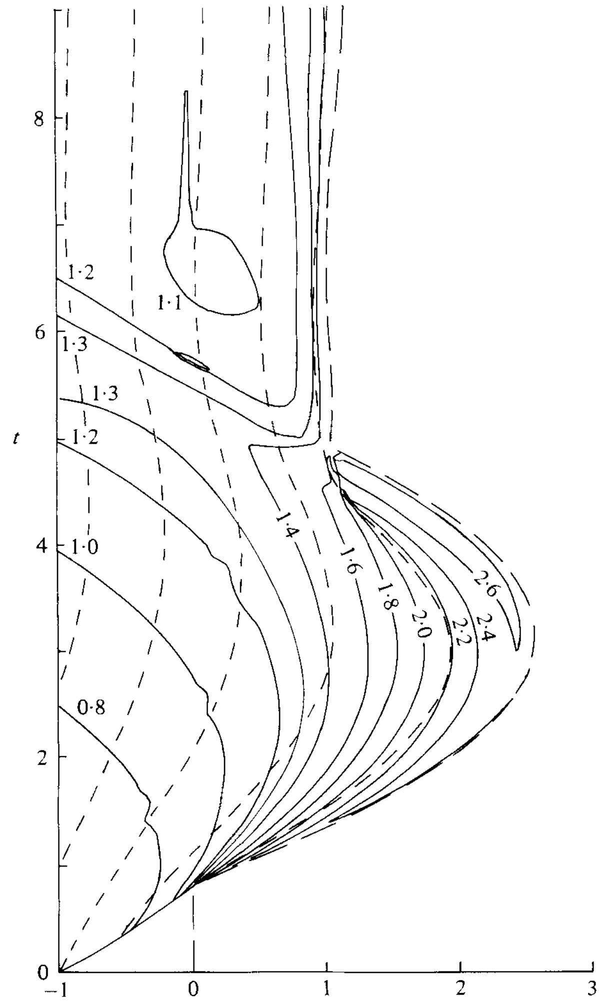

FIGURE 3. ——, contours of longshore velocity v, at values shown; ---, C^0 characteristics; and ——, path of shoreline, in (x,t)-space, for a single bore of initial height 0·5.

Conversely, no information about longshore motion generated by the bore's travel up the beach is transmitted to sea.

Contours of longshore velocity are also shown. Close to the shoreline, during run-up, they are almost parabolic, due to the dominance of gravity here. A small disturbance is visible, travelling along C^0_B. This is a numerical effect, typical of Lax–Wendroff schemes, and is due to the discontinuity in beach slope at $x = -1$.

4. Periodic bores

4.1. *The onshore problem*

As an attempt to model more realistically longshore motion on a beach, we now find solutions to the equations of motion (11), (12) driven by a periodic succession of incident bores. The formulation of the problem is similar to that already used: we apply appropriate boundary conditions at $x = -1$ and solve, in $x > -1$, (12) for

$v(x, t)$, with known onshore motion, $u(x, t)$ and $h(x, t)$. The solution method adopted for the onshore problem is that of Packwood (1980), who solves (11) using a Lax–Wendroff finite-difference scheme similar to that of Hibberd & Peregrine (1979). Packwood forces periodic solutions by applying periodic boundary conditions, in the form of a periodic distribution of $\alpha_1(t) = u(-1, t) + 2c(-1, t)$, the value of the Riemann invariant on the advancing C^+ characteristics. The flow is started from rest at $t = 0$ and the computation proceeds until an effectively periodic flow throughout $x > -1$ is found. The physical variables u and h are not periodic at $x = -1$ until this is achieved.

The distribution $\alpha_1(t)$ takes a maximum value α_1^+ immediately behind an arriving bore and a minimum α_1^- ahead of it; since $\alpha_1 = 2$ is the value in the still water in $x > -1$ at $t = 0$, this value is taken to be the mean of α_1^+ and α_1^-, with a linear variation between them:

$$\alpha_1(t) = \alpha_1^+ + 2(2-\alpha_1^+)\frac{t}{T},$$

where T is the period. The value α_1^+ is determined by the height of the first bore.

The resulting onshore motion is found by Packwood (1980) both for inviscid flow and using Chézy friction. He finds, however, that the inclusion of friction has very little effect, except in determining the position of the shoreline, particularly during backwash: the effect of the inclusion of the viscous term is to leave a thin layer of water slowly draining down the beach. This is not the case for the longshore problem, where friction is important in determining the magnitude of the solutions. For these reasons, and since it facilitates the treatment of the longshore problem, we shall neglect the onshore component of bottom friction in the solutions described below; we show below that the effect of this neglect is small.

4.2. *The longshore problem*

The decoupling of the longshore problem allows us to reach some conclusions about periodic solutions for $v(x, t)$ of (12). We will show that solutions exist for any non-zero value of the friction coefficient f, and that a periodic solution $v(x, t)$ may be found directly, without having to start the computation from rest, so long as periodic values $h(x, t)$ and $u(x, t)$ are known.

Figure 4 is a sketch of such a periodic solution, showing C^0 characteristics, along with (15) holds; this may be written

$$\frac{d\gamma}{d\sigma} + f\frac{|u|\gamma}{h} = -f\frac{|u|}{h}(h + x + \tfrac{1}{2}u^2), \tag{23}$$

having neglected the x-component of friction stress, as noted above.

We use suffixes 0, 1 and 2 to denote values, on a particular characteristic, respectively immediately behind a bore and immediately ahead of and behind the succeeding bore, as shown. If the motion is periodic $\gamma_0 = \gamma_2$. The bore relations (13) may be used to relate γ_1 and γ_2: we may manipulate (13) to find

$$[\gamma] = \frac{\frac{1}{2}[hu][h^2]}{h_1 h_2 [u]} - [h + \tfrac{1}{2}u^2], \tag{24}$$

where now $[\quad] = (\quad)_1 - (\quad)_2$. The right-hand side of (24) is, in general, non-zero, so that there is a jump in γ across a bore. The role of friction is to dissipate γ along a characteristic, according to (23); periodicity is achieved when, for each characteristic,

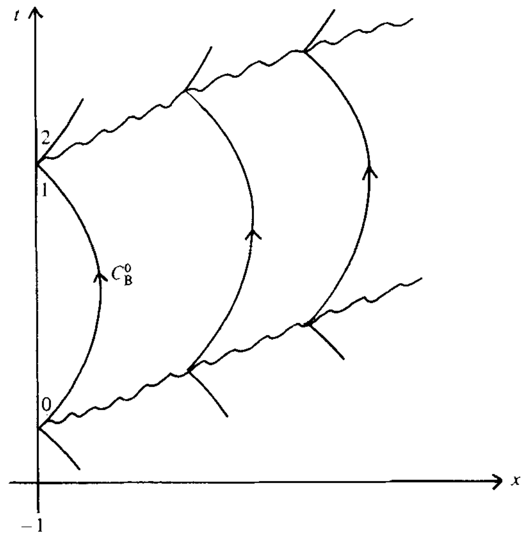

FIGURE 4. Sketch of C^0 characteristics, in (x, t)-space, for periodic motion with periodic incident bores. Paths of bores are shown by wavy lines.

the increase in γ due to a bore crossing it once per period is exactly matched by the decrease along it due to friction during a period.

We use (23) to find an explicit solution for the longshore velocity by integration along a characteristic. Equation (23) may be solved to give

$$\gamma(\sigma) = E(\sigma)\left\{\int_{\sigma_0}^{\sigma} \frac{-f|u|\,(h+\frac{1}{2}u^2+x)}{hE(\sigma')}\,d\sigma' + \gamma(\sigma_0)\right\}, \tag{25}$$

where $\sigma = \sigma_0$ denotes, as above, the point on a characteristic immediately behind a bore, and

$$E(\sigma) = \exp\left\{-\int_{\sigma_0}^{\sigma} f\frac{|u|}{h}\,d\sigma'\right\}.$$

The correct periodicity of (25) is enforced by choosing $\gamma(\sigma_0)$ to ensure that

$$\gamma(\sigma_0) - \gamma(\sigma_1) = [\gamma].$$

We find, using (25), that

$$\gamma(\sigma_0) = \frac{[\gamma] - E(\sigma_1)\displaystyle\int_{\sigma_0}^{\sigma_1} \frac{f\frac{|u|}{h}(h+\frac{1}{2}u^2+x)}{E(\sigma)}\,d\sigma}{1-E(\sigma_1)}. \tag{26}$$

The effect of a small value of f may be seen from this solution: analysis of (25) shows that $v = O(1/f)$ as $f \to 0$.

That v becomes large if f is small is important in that v was assumed to be at most $O(1)$ when terms in ϵ^2 and ϵ^3 were neglected in (8)–(10). If v is large, these terms may also be large, depending on the magnitude of ϵ^2/f. Below, we estimate the limit on θ which this restriction imposes.

Boundary conditions at $x = -1$ may be given by specifying a distribution $\gamma_1(-1, t)$, for values of t at which $u(-1, t) > 0$; if $u(-1, t) < 0$, C^0 characteristics are crossing $x = -1$ seawards, and the value of γ_1 there is determined by the flow. A chosen

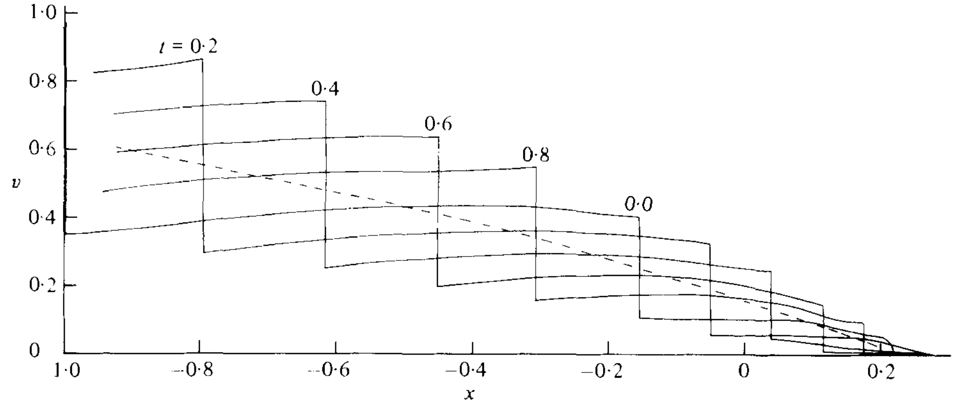

FIGURE 5. Instantaneous profiles of longshore velocity v versus x at various times during a periodic, for periodic bores. Period = 1·048; Friction coefficient f = 0·5. ----, time mean of v over a period.

distribution affects the solution only seaward of C^0_B: in the example shown below, the region between $x = -1$ and C^0_B was only a small part of the computation and no solutions here are shown.

An example of a solution, with a period $T = 1{\cdot}048$ in dimensionless units, is shown in figure 5. The initial bore has a height of 0·5, and the onshore motion settled to effective periodicity after about 12 periods. At this value of T, no secondary bores form within the flow, and no reflected bores would be formed offshore of $x = -1$.

Full solutions are shown for $f = 0{\cdot}5$: in terms of dimensional quantities, this could correspond to, for instance, waves of period 10 s, breaking at a depth of 1 m, on a beach of slope 1:30, with a friction coefficient $C = 0{\cdot}017$.

Difficulties were found in computing the solution near the shoreline, where depth and bore height are both small, and the bore width, as described numerically, is no longer small. The computed values of v here are probably less accurate than elsewhere.

Figure 5 shows instantaneous profiles of longshore velocity v at intervals during a period: $t = 0$ is the moment at which a bore arrives at $x = -1$. The mean-velocity profile $\bar{v}(x)$, averaged over a period, is also shown.

As a bore nears the shoreline, its height gradually decreases: this correspondingly decreases the driving stress for the longshore velocity, which is indeed zero at the shoreline. This is due also to the effect of friction in shallow water: we may see from (23) that frictional damping increases as h decreases. That the longshore current is zero at the shoreline agrees with the findings of experiments such as those of Galvin & Eagleson (1965) and confirms that the use of this as a boundary condition in simpler descriptions of longshore motion, such as that of Longuet-Higgins (1970*a*,*b*), is reasonable.

Figure 6 shows contours of longshore velocity and paths of C^0 characteristics for one period.

Since by changing the value of f we do not change the paths of the characteristics but only the values of v found along them, we do not show full results for other values of f. In figure 7, however, we show, for various values of f, envelopes of maximum and minimum values of v, over one period, at all points across the surf zone. The details of velocity profiles within these envelopes are similar to those of figure 5. For each value of f, $\bar{v}$ varies almost linearly with x: this agrees qualitatively with the results of Longuet-Higgins (1970*a*,*b*). He finds, using linear long-wave theory to describe wave motion, and before considering the effects of offshore mixing, that longshore

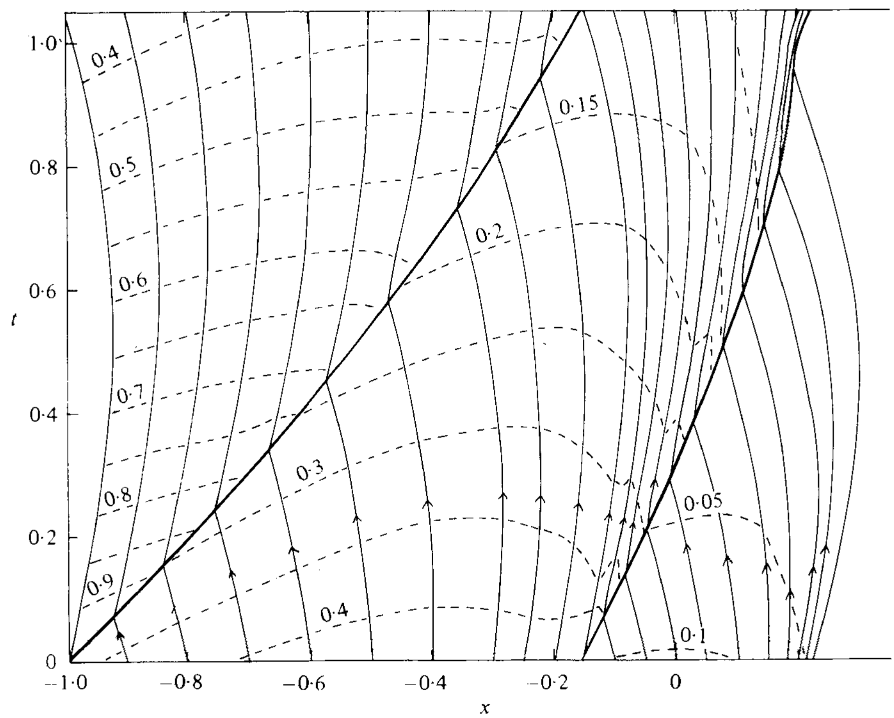

FIGURE 6. ——, C^0 characteristics, and ---, contours of longshore velocity, at values shown, for periodic incident bores. Period = 1·048; friction coefficient $f = 0{\cdot}5$. The path of the bore is shown by a heavy line.

current decreases linearly from a maximum $\bar{v}_B$ at the breaker line to zero at the shoreline. In our notation

$$\bar{v}_B = \frac{5\pi}{8}\frac{\alpha}{f}, \tag{27}$$

where α is the ratio of amplitude to depth, assumed by Longuet-Higgins to remain constant across the surf zone. In the example already used here, this ratio is 0·29 at the seaward boundary, increasing to about 0·4 midway through the surf zone. We show a comparison of our results with those of Longuet-Higgins in figure 7, using the value $\alpha = 0{\cdot}29$ in (27).

The difference between the results of Longuet-Higgins (1970*a*,*b*) and the present work is due firstly to modelling of frictional dissipation which differs from the analysis of Longuet-Higgins who assumed sinusoidal onshore motion, and secondly to the difference in the incident longshore momentum fluxes of linear waves and of bores. The first of these seems to be the greater effect: choosing $\alpha = 0{\cdot}24$ in order to match the radiation stresses S_{xy} of the two models, thereby removing the second effect, reduces $\bar{v}_B$ found from (27) from 1·14 to 0·95 for $f = 0{\cdot}5$, while the value found in the present work is about $\bar{v} = 0{\cdot}64$.

We may now examine the strength of the restriction on the validity of the solutions presented here of the assumption of a small angle of incidence of the approaching waves. If we assume that the second and third terms on the left-hand side of (10) are not larger than the first term, then the neglect of the right-hand side is justified, on averaging over a wave period, and remembering that $\epsilon = c^{-1}\sin\theta$, where c is

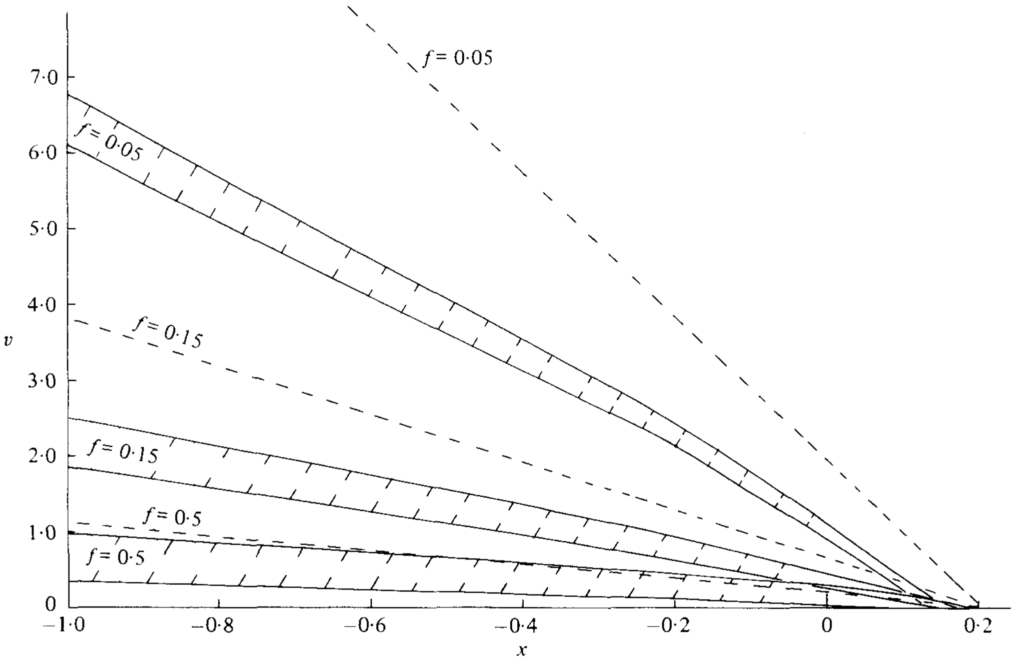

FIGURE 7. Envelope of maximum and minimum values over one period of longshore velocity v for various values of friction coefficient f. ——, solution of Longuet-Higgins (1970*a*, *b*) as described in the text.

the phase speed of the incident waves, if

$$\sin^2\theta \ll \frac{c^2}{\bar{v}}, \tag{28a}$$

$$\sin^2\theta \ll \frac{fTU_m c^2}{\bar{h}[v]}, \tag{28b}$$

where an overbar denotes the average over a period, $U_m = \overline{|u|}$ and we have assumed that

$$\overline{\left(\frac{|u|\,v}{h}\right)} \approx \frac{\overline{|u|}\,v}{h}.$$

The first of the conditions (28) also ensures that the neglect of the right-hand sides of (8) and (9) is justified.

On applying these restrictions to the calculated example, with $f = 0{\cdot}5$, we find the restrictions on the angle of incidence to be

$$\sin\theta \ll 1{\cdot}6, \quad \sin\theta \ll 0{\cdot}5.$$

In practice such restrictions are unlikely to be severe. The effect of depth refraction on waves approaching a coast ensures that in most circumstances, and particularly for gentle offshore waves, the angle of incidence of waves when they break is indeed small.

We may use the results found here retrospectively to confirm that the effect of onshore friction on the longshore problem is indeed small, so that the neglect of the second term of (15), as described above, is justified. By taking the approximate average over time of the equations of motion, we find that the inclusion of the neglected term changes $\bar{v}$ by only about 3% when $f = 0{\cdot}5$, and by less when f is smaller.

Comparison with experimental and field measurements of longshore currents is hampered by the effect of random processes in the surf zone. Longuet-Higgins (1970*a*, *b*) supposes that theoretical longshore current profiles are modified by horizontal turbulent mixing, which carries some longshore momentum seaward across the breaker line. He finds rough quantitative agreement with the results of the experiments of Galvin & Eagleson (1965) by assuming that local distance from the shoreline is an appropriate mixing length. However, since turbulence is induced by the passage of turbulent bores across the shoreline, it may be that local bore width is a more realistic local mixing length. Smoothing of the longshore current profile may also be due to randomness in the incident wave field: Battjes (1972) shows that the effect of this on the current profile may be similar to the effect of lateral mixing.

4.3. *Development of periodic solutions*

If its angle of incidence is not so large as to violate the conditions (28), then a periodic wavetrain may generate steady longshore motion on a beach as described above. However, some time is required for such a solution to develop from rest (or from any other initial state) and, in practice, the incident waves themselves may also vary with time. The periodic solutions already described are, therefore, likely to be appropriate only if the time taken to reach such a solution is not too long, when compared with the timescale of variations in the incident waves.

Inspection of the longshore momentum equation, (12), shows that the use of ft as a time variable and fx as a space variable in solving this equation gives solutions independent of f. However, the onshore momentum equation cannot so be treated: the term arising from the bottom slope cannot be scaled. We cannot, therefore, completely remove f from the problem, but this suggests that the use of fT as a timescale may be appropriate in studying longshore motion. Note that this quantity is independent of beach slope: in dimensional quantities

$$fT = CT^*\left(\frac{g^*}{H^*}\right)^{\frac{1}{2}}.$$

This effect also means that the timescale of the development of longshore motion is inversely proportional to the friction factor.

As a specific example of this, and for simplicity in formulating the problem, we shall consider the development of periodic solutions in terms of the quantity $\gamma_s(\sigma)$, defined as $\gamma(x,t)$ evaluated along C^0_s, the C^0 characteristic that leaves $x = -1$ when the initial bore arrives there, at $t = 0$; σ is a coordinate measured along this characteristic. As each successive bore meets C^0_s it instantaneously increases γ_s by an amount $[\gamma]$, while between bores γ_s decreases according to (23). We use the notation $\gamma_s^{(n)}$ to denote the value of γ_s at the start of the nth period.

We may now use (25) and the bore relation (24) to find

$$\gamma_s^{(n+1)} = E(\sigma_1)\,(\gamma_s^{(n)} + g_1) + [\gamma], \tag{29}$$

where

$$g_1 = -\int_{\sigma_0}^{\sigma_1} \frac{f|u|\,(h+\frac{1}{2}u^2+x)}{hE(\sigma')}\,d\sigma'.$$

We assume that $E(\sigma_1)$, g_1 and $[\gamma]$ do not vary from one wave to the next, though in practice they do vary as the periodic onshore motion develops. However, since we seek only an estimate of the timescale of the development of longshore motion, we

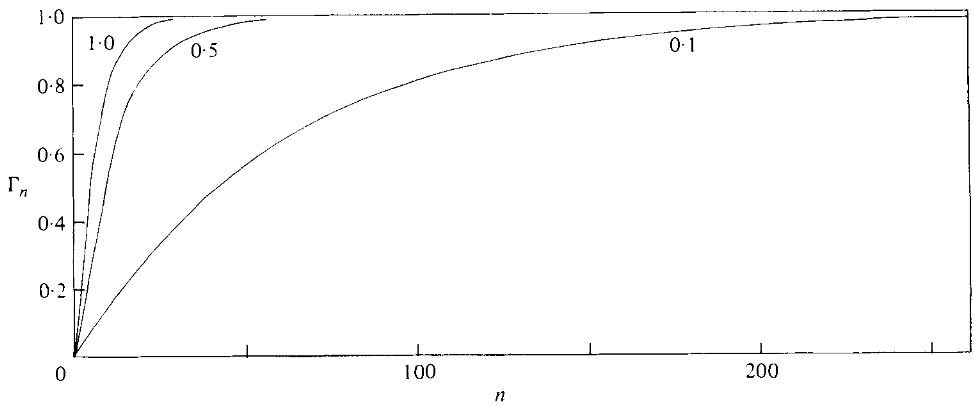

FIGURE 8. Development of periodic solutions: Γ_n versus number of periods n for values shown of friction coefficient f.

neglect this, and use below the values for $E(\sigma_1)$, g_1 and $[\gamma]$ for the fully developed periodic onshore motion, all evaluated at $x = -1$.

The periodic longshore motion already described is represented in (29) by $\gamma_s^{(\infty)}$, the solution for which $\gamma_s^{(n+1)} = \gamma_s^{(n)}$. Writing

$$\Gamma_n = \frac{\gamma_s^{(n)}}{\gamma_s^{(\infty)}}$$

we find from (29)

$$\Gamma_{n+1} = E(\sigma_1)\,\Gamma_n + 1 - E(\sigma_1),$$

which, with the initial condition $\Gamma_0 = 0$, expresses iteratively the variation with time of the quantity Γ. Γ represents the state of the development of longshore motion: $\Gamma = 0$ and $\Gamma = 1$ correspond respectively to the initial undisturbed state and the eventual periodic solution.

We show in figure 8 the variation of Γ with time for three values of f. Although Γ is only defined at integer values of $n = t/T$, we show it, for convenience, as a continuous function of n. In each case, it varies smoothly and monotonically from zero to almost one for large values of n. This suggests that the periodic solution of (25) would indeed be the result of solving the full equations from rest.

Figure 8 allows us to estimate the time required to reach periodicity; e.g. for $f = 0{\cdot}5$, with waves of period 10 s, Γ is within 5 % of its steady-state value after about 6 min, whereas for $f = 0{\cdot}1$ this takes about 30 min. Such times are likely to be too long, in many cases, to allow for the development of a periodic solution over, say, the period of an edge wave or the duration of a wave group.

5. Concluding remarks

We have described wave-induced longshore motion on a beach using a model for water motion appropriate in shallow water. The use of turbulent bores to represent wave fronts has enabled us to estimate accurately the dissipation rate within the surf zone: this is particularly important in finding longshore velocity, the driving stress for which is dependent upon the dissipation rate. The fact that longshore motion may be considered independently of the method used to find the onshore motion has proved to be of value in considering periodic solutions of the equations of motion.

The most important limitation to our work is the use of the Chézy friction law. Whereas the effect of friction on the onshore motion is, as found by Packwood (1980), significant only near the run-up tip, its effect on longshore motion is important throughout the surf zone. Moreover, although the work of Packwood & Peregrine (1981) suggests that the major flow properties of onshore motion are often insensitive to the representation of bottom friction, this is unlikely to be the case for longshore motion: the solutions of §4 are sensitive to the value of the Chézy coefficient, and the form of the averaged solution, described in the preceding paragraph, reflects the form of the Chézy law. A significant improvement on the present work would be the development of a boundary-layer model to describe bottom friction.

Although previous use of the Chézy law suggests that it provides a useful description of the effect of bottom friction on major flow properties, care may be needed in its application to problems involving longshore motion in which the value of the bottom shear stress is required. Packwood & Peregrine (1981) find that the value of bottom shear stress predicted by the Chézy law may be very different from those found by the boundary-layer model of Packwood (1980), although major flow properties predicted by the two models may otherwise be very similar.

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